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Multiple addition theorem for discrete and continuous nonlinear problems

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Abstract
The addition relation for the Riemann theta functions and for its limits, which leads to the appearance of exponential functions in soliton type equations is discussed. The form of addition property presented resolves itself to the factorization of the N-tuple product of the shifted functions and it seems to be useful for analysis of soliton type continuous and discrete processes in the $N + 1$ space–time. A close relation with the natural generalization of bi- and tri-linear operators into multiple linear operators concludes the paper.

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1. Introduction

The main goal of this paper is the presentation of the role of the addition property (AP), its relation to the famous bilinear operator formalism and its universality, since using the AP, the quasi-periodic and soliton processes can be considered in an identical manner. The generalization of the standard version of the AP to the version which can be linked with multilinear operators is a constructive step in this direction. It seems that this generalized version can be useful in the case of multi-dimensional soliton type problems.

There is an opinion that the huge success of the bilinear formalism in soliton theory can be related to the AP for τ-functions which appear in the majority of soliton equations:

$$\tau(z + w)\,\tau(z - w) = \sum_{\varepsilon} W_{\varepsilon}(w)\, Z_{\varepsilon}(z) \tag{1}$$

where $z = \kappa x + \omega t \in C^g$, $\varepsilon \in Z_2^g$; $\tau : C^g \to C$ and $W_\varepsilon(w)$, $Z_\varepsilon(z) : C^g \times Z^g \to C$.

The essential point here is the factorization of the right-hand side of (1), in which the functions W_ε and Z_ε depend on w and z, respectively and exclusively. There are a few versions of the AP, according to scheme (1) [1–3], and the factorization appears in each one.

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In applications to the soliton type equations, the argument $\boldsymbol{z}$ usually depends on space and time, while $\boldsymbol{w}$ plays the role of a fixed constant parameter. In a few papers [4] it was shown that (1) allows one, in a straightforward manner, to determine derivatives of logarithms of the τ-function, which are useful in the differential version of soliton type equations. For the discrete soliton type equations the form (1) has a direct and immediate application.

As shown in the cited references, a class containing exponential functions as well as the Riemann theta functions has just the AP according to (1).

In order to illustrate an application of the AP, we present below two examples: the discrete Hirota equation and doubly discrete sine-Gordon equation (dd-sGe). In the limit, when the step tends to zero, the first Hirota equation has a trivial limit, while the second one (dd-sGe) becomes a standard sGe.

2. The Hirota equation

As an elementary example of the AP we present the system of dispersion equations for the functional (discrete) equations

$$\begin{aligned} &a\tau\,(x+h,y,t)\,\tau\,(x-h,y,t)+b\tau\,(x,y+h,t)\,\tau\,(x,y-h,t)\\ &\qquad +c\tau\,(x,y,t+h)\,\tau\,(x,y,t-h)=C\tau^2\,(x,y,t) \end{aligned} \tag{2}$$

where h is the step and a, b, c, C are constants. When $C = 0$ this equation is known as the Hirota equation and then one can find its soliton solutions as in [5]. Quasiperiodic solutions are reported in [6] and another class of solutions in [7]. In bilinear operator language (2) can be written as

$$[a\exp\,(hD_x)+b\exp(hD_y)+c\exp\,(hD_t)]\,(\tau\circ\tau)=C\tau^2 \tag{3}$$

where the bilinear operator D_x (in the scalar version) is defined as

$$\begin{aligned} (D_x)^n\,(\tau\circ\tau):\ &=(\partial_x-\partial_{x_1})^n\tau\,(x,y,t)\,\tau\,(x_1,y,t)\,|_{x_1=x}\\ &=(\partial_s)^n\,[\tau\,(x+s,y,t)\,\tau\,(x-s,y,t)]\,|_{s=0}. \end{aligned} \tag{4}$$

Assuming that the τ-function argument is $\boldsymbol{z} = \boldsymbol{k}x + \boldsymbol{l}y + \boldsymbol{w}t \in C^g$, equation (2) can be rewritten as

$$a\tau\,(\boldsymbol{z}+\boldsymbol{k}h)\,\tau\,(\boldsymbol{z}-\boldsymbol{k}h)+b\tau\,(\boldsymbol{z}+\boldsymbol{l}h)\,\tau\,(\boldsymbol{z}-\boldsymbol{l}h)+c\tau\,(\boldsymbol{z}+\boldsymbol{w}h)\,\tau\,(\boldsymbol{z}-\boldsymbol{w}h)=C\tau^2\,(\boldsymbol{z}) \tag{5}$$

which is just suitable for the application of the AP. We obtain the functional equation

$$\sum_{\varepsilon\in Z_2^g}[aW_\varepsilon\,(\boldsymbol{k}h)+bW_\varepsilon\,(\boldsymbol{l}h)+cW_\varepsilon\,(\boldsymbol{w}h)-CW_\varepsilon\,(0)]\,Z_\varepsilon\,(\boldsymbol{z})=0 \tag{6}$$

which in the case of independent functions $Z_\varepsilon(z)$, $\varepsilon \in Z_2^g$ leads to the system of algebraic equations

$$aW_\varepsilon\,(\boldsymbol{k}h)+bW_\varepsilon\,(\boldsymbol{l}h)+cW_\varepsilon\,(\boldsymbol{w}h)-CW_\varepsilon\,(0)=0 \qquad \text{for each} \quad \boldsymbol{\varepsilon}\in Z_2^g. \tag{7}$$

For the fixed type of solution (which determines the class of functions W_ε) and for a fixed step h, equations (7) determine the relations between a, b, c, C and $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{w}$. Thus the dispersion equations (7) are valid for both soliton and quasiperiodic processes, and even for processes in the form of solitons on the periodic background.

3. The discrete sine-Gordon equation

We consider the functional (discrete–discrete) version of the sine-Gordon equation (dd-sGe) in the form

$$\frac{1}{h^2}\sin\left(\frac{u^{++}+u^{--}-u^{+-}-u^{-+}}{4}\right)=\sin\left(\frac{u^{++}+u^{--}+u^{+-}+u^{-+}}{4}\right) \tag{8}$$

where $u^{\pm\pm} := u\,(\xi \pm h, \tau \pm h)$. It is obvious that in the limit $h \to 0$ equation (8) becomes the traditional sGe in light cone coordinates

$$u_{,\xi\tau} = \sin u. \tag{9}$$

We look for the quasi-periodic solutions (8) in the form, which is identical (up to constant parameters) with solutions of (9),

$$u\,(\xi, \tau) = 2\mathrm{i} \ln \frac{\theta(\boldsymbol{z} + \boldsymbol{e}/2|B)}{\theta\,(\boldsymbol{z}|B)} \tag{10}$$

where $\theta\,(\boldsymbol{z}|B)$ denotes the Riemann theta function of argument $\boldsymbol{z} = \boldsymbol{k}\xi + \boldsymbol{u}\tau \in \boldsymbol{C}^g$, parametrized by the Riemannian matrix $B \in \boldsymbol{C}^{g\times g}$; $\boldsymbol{e} = [1, \ldots, 1] \in Z^g$, see e.g. [1,4]. Moreover, in order to obtain the real solutions we require $\theta\left(\boldsymbol{z} + \frac{\varepsilon}{2}|B\right) = [\theta\,(\boldsymbol{z}|B)]^*$.

Since

$$u\,(\xi + h, \tau \pm h) = 2\mathrm{i} \ln \frac{\theta\,(\boldsymbol{z} + (\boldsymbol{k} \pm \boldsymbol{u})\,h + \boldsymbol{e}/2|B)}{\theta(\boldsymbol{z} + (\boldsymbol{k} \pm \boldsymbol{u})\,h|B)} = 2\mathrm{i} \ln \frac{\theta(\boldsymbol{z} + \boldsymbol{w}_\pm + \boldsymbol{e}/2|B)}{\theta(\boldsymbol{z} + \boldsymbol{w}_\pm|B)} \tag{11}$$

and similar relations hold for $u\,(\xi - h, \tau \pm h)$, after simple manipulations we can rewrite (8) in the form

$$\begin{aligned} &\frac{\theta\,(\boldsymbol{z} + \boldsymbol{w}_-)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_-) - h^2\theta\,(\boldsymbol{z} + \boldsymbol{w}_- + \boldsymbol{e}/2)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_- + \boldsymbol{e}/2)}{\theta\,(\boldsymbol{z} + \boldsymbol{w}_+)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_+)} \\ &\qquad = \frac{\theta\,(\boldsymbol{z} + \boldsymbol{w}_- + \boldsymbol{e}/2)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_- + \boldsymbol{e}/2) - h^2\theta\,(\boldsymbol{z} + \boldsymbol{w}_-)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_-)}{\theta\,(\boldsymbol{z} + \boldsymbol{w}_+ + \boldsymbol{e}/2)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_+ + \boldsymbol{e}/2)} \end{aligned} \tag{12}$$

where $\boldsymbol{w}_+ = (\boldsymbol{k} + \boldsymbol{u})\,h,\ \boldsymbol{w}_- = (\boldsymbol{k} - \boldsymbol{u})\,h$. Let us assume that these quotients are constant $(= C + 1)$ for arbitrary $\boldsymbol{z}$. Because of a general property $\theta\,(\boldsymbol{z} + \boldsymbol{e}/2) = \theta\,(\boldsymbol{z} - \boldsymbol{e}/2)$, both quotients lead to the same result:

$$\begin{aligned} &\theta\,(\boldsymbol{z} + \boldsymbol{w}_-)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_-) - h^2\theta\,(\boldsymbol{z} + \boldsymbol{w}_- + \boldsymbol{e}/2)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}_- + \boldsymbol{e}/2) \\ &\qquad = (C + 1)\,\theta\,(\boldsymbol{z} + \boldsymbol{w}_+)\,\theta(\boldsymbol{z} - \boldsymbol{w}_+). \end{aligned} \tag{13}$$

Now the AP can be applied. The Riemann theta functions do have the AP

$$\theta\,(\boldsymbol{z} + \boldsymbol{w}|B)\,\theta\,(\boldsymbol{z} - \boldsymbol{w}|B) = \sum_{\varepsilon \in Z_2^g} W_\varepsilon\,(\boldsymbol{w})\ \theta^2\left(\boldsymbol{z} + \frac{\varepsilon}{2}\middle|\, B\right) \tag{14}$$

for $\boldsymbol{z},\ \boldsymbol{w} \in C^g$. W_ε coefficients can be expressed by theta-constants [4], but—and it is important—do not depend on ξ and τ. Then equation (13) can be written as

$$\begin{aligned} &\sum_{\varepsilon \in Z_2^g} W_\varepsilon\,(\boldsymbol{w}_-)\,\theta^2\left(\boldsymbol{z} + \frac{\varepsilon}{2}\middle|\, B\right) - h^2 \sum_{\varepsilon \in Z_2^g} W_\varepsilon\,(\boldsymbol{w}_-)\,\theta^2\left(\boldsymbol{z} + \left(\frac{\boldsymbol{e} + \varepsilon}{2}\right)\middle|\, B\right) \\ &\qquad = (C + 1) \sum_{\varepsilon \in Z_2^g} W_\varepsilon\,(\boldsymbol{w}_+)\,\theta^2\left(\boldsymbol{z} + \frac{\varepsilon}{2}\middle|\, B\right) \end{aligned} \tag{15}$$

or as a simple functional equation

$$\sum_{\varepsilon \in Z_2^g} [W_\varepsilon\,(\boldsymbol{w}_-) - h^2 W_{e-\varepsilon}\,(\boldsymbol{w}_-) - (C + 1)\,W_\varepsilon\,(\boldsymbol{w}_+)]\theta^2\left(\boldsymbol{z} + \frac{\varepsilon}{2}\middle|\, B\right) = 0. \tag{16}$$

Since $\theta^2\left(\boldsymbol{z} + \frac{\varepsilon}{2}|B\right)$ labelled by $\varepsilon \in Z_2^g$ form a set of linearly independent functions, we finally arrive at the requirement that for any $\varepsilon \in Z_2^g$

$$W_\varepsilon\,(\boldsymbol{w}_-) - h^2 W_{e-\varepsilon}\,(\boldsymbol{w}_-) - (C + 1)\,W_\varepsilon\,(\boldsymbol{w}_+) = 0. \tag{17}$$

Equations (17) determine $\boldsymbol{k}$ and $\boldsymbol{u}$, (and C) and these are linearly related to the propagation vectors $\boldsymbol{\kappa}$ and angular frequencies $\boldsymbol{\omega}$ in the laboratory coordinate system $(\boldsymbol{z} = \boldsymbol{\kappa}x + \boldsymbol{\omega}t)$.

Therefore these equations represent a system of dispersion equations for the discussed dd-sGe. Nontrivial solutions of (17) determine the solutions of the starting equation (8), but for the higher g, since then the system is overdetermined, also some and even all elements of the matrix B.

As (dd-sGe)$\to$(sGe), with $h \to 0$, (17) also tends to the dispersion equation for standard sGe:

$$\sum_{i,j} k_i u_j W_{\varepsilon,ij} + \tfrac{1}{2}\left(\delta_{e,\varepsilon} - c\delta_{\varepsilon,0}\right) = 0. \tag{18}$$

where $W_{\varepsilon,ij} := \frac{\partial^2}{\partial w_i \partial w_j} W_\varepsilon(\boldsymbol{w})\,|_{w=0}$. In order to prove this statement one can substitute a new constant $c = -C/h^2$ and use the relations $W_\varepsilon(\boldsymbol{z}) = W_\varepsilon(-\boldsymbol{z})$, $W_\varepsilon(0) = \delta_{\varepsilon,0}$, where δ represents the Kronecker symbol.

4. Tri-linear operator

In order to extend direct methods in the spirit of the Hirota bilinear formalism to a broader class of equations, the tri-linear operators T and T^* were introduced [8]:

$$(T)^n\,(\tau \circ \tau \circ \tau) := \left(\partial_z + j\partial_{w_1} + j^2\partial_{w_2}\right)^n \tau(z)\,\tau(w_1)\,\tau(w_2)\,|_{w_2=w_1=z} \tag{19}$$

$$(T^*)^n\,(\tau \circ \tau \circ \tau) := \left(\partial_z + j^2\partial_{w_1} + j\partial_{w_2}\right)^n \tau(z)\,\tau(w_1)\,\tau(w_2)\,|_{w_2=w_1=z} \tag{20}$$

where $j = \exp(\mathrm{i}2\pi/3)$.

In this language, for example, the fifth-order equation of the Lax hierarchy

$$u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x + u_t = 0 \tag{21}$$

although lacking a bilinear representation, can be written in tri-linear form [9]

$$\left(7T_x^6 + 20T_x^3 T_x^{*3} + 27T_x T_t\right) F \circ F \circ F = 0 \tag{22}$$

where $u = 2(\ln F)_{xx}$. The soliton solutions of this equation were also discussed in [10], and we will return to this equation in the last paragraph of this paper. However, first let us try to generalize the concept of bi- and tri-linear operators.

5. Multiple addition theorem for theta functions

One can introduce a multilinear (J-linear) operator by the relation

$$(T)^n\,(\tau \circ \cdots \circ \tau) := \left[\left(\partial_{z_0} + j\partial_{z_1} + \cdots + j^{J-1}\partial_{z_{J-1}}\right)^n \prod_{i=0}^{J-1} \tau(z_i)\right]_{z_{J-1}=\cdots=z_1=z_0=z} \tag{23}$$

where $j = \exp(\mathrm{i}2\pi/J)$ and symbolically $(\tau \circ \cdots \circ \tau) = (\tau\circ)^J$.

We are convinced that the effectiveness of the bi- and tri-linear operators formalism depends on the relevant AP of the functions to which this formalism is applied. Therefore, the fundamental question is which class of functions has the multiple AP. Instead of the class of exponential functions appearing in the solutions of soliton type equations we focus our attention on the Riemann theta functions constituting a more general class of functions and expressing the quasi-periodic solutions. Obviously, exponential functions can be considered as the limiting case of the theta functions.

Following [1–4], we adopt the definition of the Riemann theta function as

$$\theta(\boldsymbol{z}|B) = \sum_{n\in\mathbb{Z}^g} \exp\left[\mathrm{i}\pi\left(2\langle \boldsymbol{z}, \boldsymbol{n}\rangle + \langle \boldsymbol{n}, B\boldsymbol{n}\rangle\right)\right] \tag{24}$$

where $z \in C^g$, $B \in C^{g\times g}$ is the Riemann matrix, (i.e. symmetric with a positively defined imaginary part) and $\langle z, n\rangle := \sum_{j=1}^{g} z_j n_j$.

If $z, u^{(k)} \in C^g$, $k = 0, \ldots, J-1$, and

$$\sum_{k=0}^{J-1} u^{(k)} = 0 \tag{25}$$

one can prove [11] that

$$\theta\big(z + u^{(0)}|B\big)\,\theta\big(z + u^{(1)}|B\big)\,\theta\big(z + u^{(2)}|B\big)\cdots\theta\big(z + u^{(J-1)}|B\big) = \sum_{\varepsilon\in\mathbb{Z}_J^g} W_\varepsilon\big(u^{(0)}, \ldots, u^{(J-1)}\big)\, Z_\varepsilon\,(z) \tag{26}$$

where

$$Z_\varepsilon\,(z) = \exp\left[\mathrm{i}\pi\,(2\,\langle z, \varepsilon\rangle + \langle \varepsilon, B\varepsilon\rangle)\right]\;\theta\,(Jz + B\varepsilon|JB) \tag{27}$$

$$W_\varepsilon\big(u^{(0)}, \ldots, u^{(J-1)}\big) = \exp\big(\mathrm{i}2\pi\,\langle u^{(0)}, \varepsilon\rangle\big)\,\theta\left(\begin{matrix} u^{(0)} - u^{(1)} + B\varepsilon \\ u^{(0)} - u^{(2)} + B\varepsilon \\ \cdots \\ u^{(0)} - u^{(J-1)} + B\varepsilon \end{matrix}\;\middle\|\;\begin{bmatrix} 2B & B & \cdot\cdot & B \\ B & 2B & \cdot\cdot & B \\ \cdot\cdot & \cdot\cdot & 2B & \cdot\cdot \\ B & B & \cdot\cdot & 2B \end{bmatrix}\right). \tag{28}$$

All theta functions are of order g, except that appearing in (28), which is of order $(J-1)\,g$. The sum in (26) is over $\varepsilon \in \mathbb{Z}_J^g$, i.e. over g-dimensional vectors whose components are $0, 1, \ldots, J-1$, and therefore the sum contains J^g elements. Equation (26), written here for theta functions, is a natural generalization of the AP (1). The same form has the generalized AP for exponential functions defined by

$$E\big(z|\tilde{B}\big) = \sum_{n\in\mathbb{Z}_2^g} \exp[\mathrm{i}\pi(2\langle z, n\rangle + \langle n, \tilde{B}n\rangle)] \tag{29}$$

which appear in the solutions of standard soliton equations. (Observe that the difference between (24) and (29) is only in the number of elements in the sum.) It is convenient to assume that diagonal elements of matrix $\tilde{B} \in C^{g\times g}$ are real. The constraint (25) can be eliminated easily by introducing new parameters $w^{(1)}, \ldots, w^{(J-1)}$ instead of $u^{(0)}, u^{(1)}, \ldots, u^{(J-1)}$ according to the relation

$$w^{(k)} = \frac{1}{j-1}\big(u^{(k)} - ju^{(k+1)}\left(1 - \delta_{k,J-1}\right) - ju^{(1)}\delta_{k,J-1}\big) \qquad k = 1, \ldots, J-1 \tag{30}$$

where $\delta_{k,J-1}$ is the standard Kronecker symbol. Inversely

$$u^{(k)} = j^{1-k}\left(j^{J-1}\sum_{m=1}^{k-1} j^m w^{(m)} + \sum_{m=1}^{J-1} j^m w^{(m)}\right) \qquad k = 1, \ldots, J-1 \tag{31}$$

$$u^{(0)} = \sum_{m=1}^{J-1} w^{(m)}. \tag{32}$$

In table 1 below we present relations between $u^{(p)}$ and $w^{(q)}$ for $J = 2$–5.

Note that the choice of w parameters is not unique. The set adopted here gives a full correspondence with tri-linear operators introduced earlier in soliton theory [8]. Since for arbitrary integer, J, the sum $\sum_{k=0}^{J-1} j^k = 0$, it is seen that the requirement (25) is fulfilled for any set $w^{(q)}$.

As already mentioned, there exist several versions of the addition theorem for theta functions. To our knowledge only one form [1, 12] leads to the product of an arbitrary number of shifted theta functions as in (26), but the right-hand side is essentially different and unusable for our purposes.

Table 1. The lowest derivatives of $W_\varepsilon(\boldsymbol{u}^{(1)})$ for $\boldsymbol{u}^{(1)} = 0$ and $J = 2$–5; see equations (34) and (35).

$J =$	2	3	4	5
$j =$	-1	$\exp(\mathrm{i}2\pi/3)$	i	$\exp(\mathrm{i}2\pi/5)$
$\boldsymbol{u}^{(0)} =$	$\boldsymbol{w}^{(1)}$	$\boldsymbol{w}^{(1)}+\boldsymbol{w}^{(2)}$	$\boldsymbol{w}^{(1)}+\boldsymbol{w}^{(2)}+\boldsymbol{w}^{(3)}$	$\boldsymbol{w}^{(1)}+\boldsymbol{w}^{(2)}+\boldsymbol{w}^{(3)}+\boldsymbol{w}^{(4)}$
$\boldsymbol{u}^{(1)} =$	$j\boldsymbol{w}^{(1)}$	$j\boldsymbol{w}^{(1)}+j^2\boldsymbol{w}^{(2)}$	$j\boldsymbol{w}^{(1)}+j^2\boldsymbol{w}^{(2)}+j^3\boldsymbol{w}^{(3)}$	$j\boldsymbol{w}^{(1)}+j^2\boldsymbol{w}^{(2)}+j^3\boldsymbol{w}^{(3)}+j^4\boldsymbol{w}^{(4)}$
$\boldsymbol{u}^{(2)} =$	—	$j^2\boldsymbol{w}^{(1)}+j\boldsymbol{w}^{(2)}$	$j^3\boldsymbol{w}^{(1)}+j\boldsymbol{w}^{(2)}+j^2\boldsymbol{w}^{(3)}$	$j^4\boldsymbol{w}^{(1)}+j\boldsymbol{w}^{(2)}+j^2\boldsymbol{w}^{(3)}+j^3\boldsymbol{w}^{(4)}$
$\boldsymbol{u}^{(3)} =$	—	—	$j^2\boldsymbol{w}^{(1)}+j^3\boldsymbol{w}^{(2)}+j\boldsymbol{w}^{(3)}$	$j^3\boldsymbol{w}^{(1)}+j^4\boldsymbol{w}^{(2)}+j\boldsymbol{w}^{(3)}+j^2\boldsymbol{w}^{(4)}$
$\boldsymbol{u}^{(4)} =$	—	—	—	$j^2\boldsymbol{w}^{(1)}+j^3\boldsymbol{w}^{(2)}+j^4\boldsymbol{w}^{(3)}+j\boldsymbol{w}^{(4)}$

Table 2. The lowest derivatives of $W_\varepsilon(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)})$ for $\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} = 0$ and $J = 2$; see equations (34) and (35).

$J = 2$		
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)}}$	$\Longleftrightarrow$	$L_{\alpha\beta}$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)}}$	$\Longleftrightarrow$	$4(3 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} + 2L_{\alpha\beta\gamma\delta})$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)} u_\zeta^{(i)} u_\mu^{(i)}}$	$\Longleftrightarrow$	$8(15 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} L_{\underline{\zeta\mu}}) + 4(15 \times L_{\underline{\alpha\beta\gamma\delta}} L_{\underline{\zeta\mu}}) + 2L_{\alpha\beta\gamma\delta\zeta\mu}$

For fixed J (26) can be rewritten as

$$\exp \sum_{j=1}^{J} \left[\ln\theta\big(\boldsymbol{z} + \boldsymbol{u}^{(j)}\big) - \ln\theta\,(\boldsymbol{z}) \right] = [\theta\,(\boldsymbol{z})]^{-J} \sum_{\varepsilon} W_\varepsilon\big(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(J-1)}\big) Z_\varepsilon\,(\boldsymbol{z})\,. \tag{33}$$

Differentiating (33) with respect to different components of vectors $\boldsymbol{u}^{(k)}$ ($k = 1, \ldots, J-1$) we obtain

$$\begin{aligned} &\frac{\partial^{p+\cdots+q}}{\big(\partial u_\alpha^{(1)}\big)^p \cdots \big(\partial u_\beta^{(l)}\big)^q} \exp\left[\sum_{k=0}^{J-1} \ln\theta\left(\boldsymbol{z} + \boldsymbol{u}^{(k)}\right) - J\ln\theta\,(\boldsymbol{z})\right] \\ &\qquad = [\theta\,(\boldsymbol{z})]^{-J} \sum_{\varepsilon} \left[\frac{\partial^{p+\cdots+q}}{\big(\partial u_\alpha^{(1)}\big)^p \cdots \big(\partial u_\beta^{(l)}\big)^q} W_\varepsilon\big(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(J-1)}\big)\right] Z_\varepsilon\,(\boldsymbol{z}) \end{aligned} \tag{34}$$

where $j = \exp(\mathrm{i}2\pi/J)$ and $\boldsymbol{u}^{(0)} = \sum_{k=1}^{J-1} \boldsymbol{u}^{(k)}$. Now changing the derivatives with respect to $\boldsymbol{u}^{(k)}$ on the left-hand side into derivatives with respect to $\boldsymbol{z}$, we can easily find the relationship between the derivatives of logarithms of theta functions (with respect to $\boldsymbol{z}$) and the derivatives of W_ε functions (with respect to $\boldsymbol{u}^{(k)}$). All relations become simpler if $\boldsymbol{u}^{(k)}$ parameters are chosen to be zero.

For $\boldsymbol{u}^{(0)} = \cdots = \boldsymbol{u}^{(J-1)} = 0$ equation (34) can be written down in a more legible form, convenient for applications:

$$K_{\alpha^p,\ldots,\beta^q}\,(\ln\theta\,(\boldsymbol{z})) = \sum_{\varepsilon} W_{\varepsilon,\alpha^p,\ldots,\beta^q}\,(\boldsymbol{0})\, \frac{Z_\varepsilon\,(\boldsymbol{z})}{[\theta\,(\boldsymbol{z})]^J} \tag{35}$$

since the left-hand side of (34) reduces to combinations of the logarithm theta derivatives $K_{\alpha^p,\ldots,\beta^q}\,(\ln\theta\,(\boldsymbol{z}))$. $W_{\varepsilon,\alpha^p,\ldots,\beta^q}\,(\boldsymbol{0})$ denotes the derivatives of the W-function taken at the point $\boldsymbol{u}^{(0)} = \cdots = \boldsymbol{u}^{(J-1)} = 0$. In that manner relation (35) determines a correspondence between derivatives of the W-function and logarithmic theta function derivatives: $W_{\varepsilon,\alpha^p,\ldots,\beta^q}\,(\boldsymbol{0}) \Longleftrightarrow K_{\alpha^p,\ldots,\beta^q}\,(\ln\theta\,(\boldsymbol{z}))$.

As an example, the lowest nontrivial derivatives of $W_\varepsilon(\boldsymbol{u}^{(1)})$ and $W_\varepsilon(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)})$, respectively (at zero), up to fifth order are reported for $J = 2$, and 3 below in tables 2 and 3. In the appendix we report the lowest nontrivial derivatives (up to sixth order) also for $J = 6$.

Table 3. The lowest derivatives of $W_\varepsilon(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)})$ for $\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} = 0$ and $J = 3$; see equations (34) and (35).

$J = 3$			
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(j)}}$	$\Longleftrightarrow$	$2L_{\alpha\beta}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)}}$	$\Longleftrightarrow$	$L_{\alpha\beta}$	
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(j)}}$	$\Longleftrightarrow$	$-L_{\alpha\beta\gamma}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)}}$	$\Longleftrightarrow$	$4(3 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} + 2L_{\alpha\beta\gamma\delta})$	
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(j)}}$	$\Longleftrightarrow$	$2(3 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} + 2L_{\alpha\beta\gamma\delta})$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(j)} u_\delta^{(j)}}$	$\Longleftrightarrow$	$4L_{\alpha\beta} L_{\gamma\delta} + 2 \times L_{\alpha\underline{\gamma}} L_{\beta\underline{\delta}} + L_{\alpha\beta\gamma\delta}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)} u_\zeta^{(j)}}$	$\Longleftrightarrow$	$-2(6 \times L_{\underline{\alpha\beta}} L_{\underline{\gamma\delta}\zeta}) - L_{\alpha\beta\gamma\delta\zeta}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)} u_\zeta^{(i)} u_\mu^{(i)}}$	$\Longleftrightarrow$	$8(15 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} L_{\underline{\zeta\mu}}) + 4(15 \times L_{\underline{\alpha\beta\gamma\delta}} L_{\underline{\zeta\mu}})$ $+2L_{\alpha\beta\gamma\delta\zeta\mu}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)} u_\zeta^{(i)} u_\mu^{(j)}}$	$\Longleftrightarrow$	$4(15 \times L_{\underline{\alpha\beta}} L_{\underline{\gamma\delta}} L_{\underline{\zeta}\mu})$ $+ 2(5 \times L_{\underline{\alpha\beta\gamma\delta}} L_{\underline{\zeta}\mu} + 10 \times L_{\underline{\alpha\beta\gamma}\mu} L_{\underline{\delta\zeta}}) + L_{\alpha\beta\gamma\delta\zeta\mu}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(i)} u_\zeta^{(j)} u_\mu^{(j)}}$	$\Longleftrightarrow$	$8(3 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} L_{\zeta\mu}) + 12(2 \times L_{\alpha\beta} L_{\gamma\underline{\zeta}} L_{\delta\underline{\mu}})$ $+ 4 \times L_{\underline{\alpha\beta\gamma\zeta}} L_{\underline{\delta}\mu} + 4 \times L_{\underline{\alpha\beta\gamma}\mu} L_{\underline{\delta}\zeta} + 4L_{\alpha\beta\gamma\delta} L_{\zeta\mu}$ $+ 2(6 \times L_{\underline{\alpha\beta}} L_{\underline{\gamma\delta}\zeta\mu}) + L_{\alpha\beta\gamma\delta\zeta\mu}$	$i \neq j$
$(W_\varepsilon)_{u_\alpha^{(i)} u_\beta^{(i)} u_\gamma^{(i)} u_\delta^{(j)} u_\zeta^{(j)} u_\mu^{(j)}}$	$\Longleftrightarrow$	$4(9 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu}) + 6 \times L_{\alpha\underline{\underline{\delta}}} L_{\underline{\beta}\underline{\underline{\xi}}} L_{\underline{\gamma}\underline{\underline{\mu}}}$ $+ 9 \times L_{\underline{\alpha}\underline{\underline{\delta}}} L_{\underline{\beta\gamma}\underline{\underline{\zeta\mu}}} + 2(6 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma}\underline{\underline{\delta\zeta\mu}}}) + 9 \times L_{\underline{\alpha\beta}\underline{\underline{\delta}}} L_{\underline{\gamma}\underline{\underline{\zeta\mu}}}$ $+ L_{\alpha\beta\gamma\delta\zeta\mu}$	$i \neq j$

The remaining derivatives of order less than six vanish. $L_{\alpha\beta} := \partial_{z_\alpha z_\beta} \ln\theta\,(\boldsymbol{z}|B)\,|_{\boldsymbol{z}=0}$ etc, and here we adopted the shorthand notation which includes all possible permutations with respect to identically underlined indices: e.g. $3 \times L_{\alpha\underline{\beta}} L_{\underline{\gamma\delta}} := L_{\alpha\beta} L_{\gamma\delta} + L_{\alpha\gamma} L_{\beta\delta} + L_{\alpha\delta} L_{\beta\gamma}$.

Some introductory applications of the above results can be found in [11].

The correspondence between the system of dispersion equations and bilinear operators reported here is quite obvious. This affinity can be extended even further. For fixed J, let us introduce a hierarchy of operators $T^{(n)}$ labelled by $n = 0, 1, \ldots, J-1$

$$\begin{aligned}\left(T^{(n)}\right)^m (\tau\circ)^{(J)} :\ &= \left(T^{(n)}\right)^m (\tau \circ \cdots \circ \tau)\\ &= \left(T^{(n)}\right)^m [\tau\,(\boldsymbol{z} + \boldsymbol{u}_0) \cdots \tau\,(\boldsymbol{z} + \boldsymbol{u}_{J-1})]|_{\boldsymbol{u}_0 = \cdots \boldsymbol{u}_{J-1} = 0}\end{aligned} \tag{36}$$

where

$$T^{(0)} = \sum_{m=0}^{J-1} \partial_{\boldsymbol{u}^{(m)}} \tag{37}$$

$$T^{(n)} = \partial_{\boldsymbol{u}^{(0)}} + \sum_{m=J-n+1}^{J-1} j^m \partial_{\boldsymbol{u}^{(m+n-J)}} + \sum_{m=1}^{J-n} j^m \partial_{\boldsymbol{u}^{(m+n-1)}} \qquad 0 \neq n < J \tag{38}$$

and differentiation relates, of course, to the indicated components of $\boldsymbol{u}^{(m)}$ vectors, i.e. $\boldsymbol{u}_\alpha^{(m)}$. For $J = 2, 3, 4$ we have table 4.

It is seen that for $J = 2$ and 3 we have the standard bi- and tri-linear operators, respectively. However, for $J > 3$, the operator $T^{(2)} \neq (T^{(1)})^*$, i.e. $T^{(2)}$ is not a complex conjugate to $T^{(1)}$. For this reason the operators $T^{(n)}$ $(n = 1, \ldots, J-1)$ for fixed J, will be called associated operators. Operator $T^{(0)}$ is introduced here only for completeness.

Now, if the τ-function from (36) has the AP, the question arises how it reflects on the $W_\varepsilon\left(\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(J-1)}\right)$ functions?

Table 4. Operators $T^{(n)}$ for $J = 2, 3, 4$.

$J = 2$	$J = 3$	$J = 4$
$T^{(0)} = \partial_{u^{(0)}} + \partial_{u^{(1)}}$	$\partial_{u^{(0)}} + \partial_{u^{(1)}} + \partial_{u^{(2)}}$	$\partial_{u^{(0)}} + \partial_{u^{(1)}} + \partial_{u^{(2)}} + \partial_{u^{(3)}}$
$T^{(1)} = \partial_{u^{(0)}} + j\partial_{u^{(1)}}$	$\partial_{u^{(0)}} + j\partial_{u^{(1)}} + j^2\partial_{u^{(2)}}$	$\partial_{u^{(0)}} + j\partial_{u^{(1)}} + j^2\partial_{u^{(2)}} + j^3\partial_{u^{(3)}}$
$T^{(2)} =$ —	$\partial_{u^{(0)}} + j^2\partial_{u^{(1)}} + j\partial_{u^{(2)}}$	$\partial_{u^{(0)}} + j^3\partial_{u^{(1)}} + j\partial_{u^{(2)}} + j^2\partial_{u^{(3)}}$
$T^{(3)} =$ —	—	$\partial_{u^{(0)}} + j^2\partial_{u^{(1)}} + j^3\partial_{u^{(2)}} + j\partial_{u^{(3)}}$

Using (31) and (32) we have $\partial_{u_\alpha^{(s)}}/\partial_{w_\alpha^{(p)}} = j^{1-s+p}\left(1 + \left(j^{J-1} - 1\right)\delta_{0<p<s}\right)$ and therefore

$$\partial_{w_\alpha^{(p)}} = \sum_{s=0}^{J-1} \frac{\partial_{u_\alpha^{(s)}}}{\partial_{w_\alpha^{(p)}}}\partial_{u_\alpha^{(s)}} = \partial_{u_\alpha^{(0)}} + j^{1+p}\sum_{s=0}^{J-1}\left[j^{-s}\left(1 + \left(j^{J-1} - 1\right)\delta_{0<p<s}\right)\right]\partial_{u_\alpha^{(s)}}$$

$$= \partial_{u_\alpha^{(0)}} + \sum_{m=J-n+1}^{J-1} j^m\partial_{u_\alpha^{(m+p-J)}} + \sum_{m=1}^{J-p} j^m\partial_{u_\alpha^{(m+p-1)}} = T_\alpha^{(p)}. \tag{39}$$

This means that if the τ-function has the AP, multilinear operators according to (36)–(38) reduce to the simple differentiation of the $W_\varepsilon\left(w^{(1)}, \ldots, w^{(J-1)}\right)$ functions with respect to their arguments. In the simplest cases of bi- and tri-linear operators this assertion allows one to immediately write the system of dispersion equations on the basis of the bi- or tri-linear approximations.

As an example, let us note the bi-linear form of the Korteweg–de Vries equation and the tri-linear form of the reduction of the self-dual Yang–Mills equation:

$$\left(D_x D_t + D_x^4\right)\tau \circ \tau = 0 \tag{40}$$

$$\left(T_x^4 T_z^* + 8T_x^3 T_z T_x^* + 9T_x^2 T_t\right)\tau \circ \tau \circ \tau = 0 \tag{41}$$

coincide with the relevant dispersion equation systems:

$$\sum_{ij}\kappa_i\omega_j W_{\varepsilon, w_i w_j} + \sum_{ijkl}\kappa_i\kappa_j\kappa_k\kappa_l W_{\varepsilon, w_i w_j w_k w_l} = CW_\varepsilon \tag{42}$$

$$\sum_{ijklm}\left(\kappa_i\kappa_j\kappa_k\kappa_l\lambda_m - 8\kappa_i\kappa_j\kappa_k\lambda_l\kappa_m\right)W_{\varepsilon, w_i w_j w_k w_l v_m} + 9\sum_{ijk}\kappa_i\kappa_j\omega_k W_{\varepsilon, w_i w_j w_k} = CW_\varepsilon \tag{43}$$

where we assumed that the arguments of τ-functions depend linearly on space and time: $z_i = \kappa_i x + \omega_i t$ and $z_i = \kappa_i x + \lambda_i y + \omega_i t$, respectively. Moreover, in the second equation $W_\varepsilon = W_\varepsilon\left(w^{(1)}, w^{(2)}\right)|_{w^{(1)}=w^{(2)}=0}$ depends on two vectors, designated for typographic reasons as w and v. In both cases, C appears as the integration constant and for soliton solutions it vanishes, while for the quasi-periodic solution it has to be determined as an additional parameter. Finally, equations (42) and (43) should hold for any $\varepsilon \in \mathbb{Z}_J^g$, i.e. the first one for $\varepsilon \in \mathbb{Z}_2^g$ and the second for $\varepsilon \in \mathbb{Z}_3^g$.

In conclusion, we expect that the generalized AP reported here might be useful for the analysis of multi-dimensional soliton equations.

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Appendix

The lowest nontrivial derivatives (up to sixth order) of $W_\varepsilon\left(w^{(1)}, \ldots, w^{(5)}\right)$ for $J = 6$ are presented in table A1.

Table A1. The lowest nontrivial derivatives (up to sixth order) of $W_\varepsilon\left(\boldsymbol{w}^{(1)},\ldots,\boldsymbol{w}^{(5)}\right)$ for $J=6$.

$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(j)}}$	$2L_{\alpha\beta}$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}}$	$L_{\alpha\beta}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}}$	$-L_{\alpha\beta\gamma}$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(j)}u_\gamma^{(k)}}$	$-L_{\alpha\beta\gamma}$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}$	$4\left(3\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}+2L_{\alpha\beta\gamma\delta}\right)$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(j)}}$	$2\left(3\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}+2L_{\alpha\beta\gamma\delta}\right)$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(j)}}$	$4L_{\alpha\beta}L_{\gamma\delta}+2\times L_{\alpha\underline{\gamma}}L_{\beta\underline{\delta}}+L_{\alpha\beta\gamma\delta}$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(k)}}$	$2L_{\alpha\beta}L_{\gamma\delta}+2\times L_{\alpha\underline{\gamma}}L_{\beta\underline{\delta}}+L_{\alpha\beta\gamma\delta}$	i, j, k—different
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(j)}u_\gamma^{(k)}u_\delta^{(l)}}$	$3\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}+L_{\alpha\beta\gamma\delta}$	i, j, k, l—different
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}u_\zeta^{(j)}}$	$-2\left(6\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\delta}\zeta}\right)-L_{\alpha\beta\gamma\delta\zeta}$	$i\neq j$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(j)}u_\zeta^{(k)}}$	$-2\left(3\times L_{\underline{\alpha\beta}}L_{\underline{\gamma}\delta\zeta}\right)-3\times L_{\underline{\alpha}\delta}L_{\underline{\beta\gamma}\zeta}$ $-3\times L_{\underline{\alpha}\zeta}L_{\underline{\beta\gamma}\delta}-L_{\alpha\beta\gamma\delta\zeta}$	$j\neq i, k\neq i$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(j)}u_\zeta^{(k)}}$	$-2\left(L_{\alpha\beta}L_{\gamma\delta\zeta}+L_{\gamma\delta}L_{\alpha\beta\zeta}\right)-2\times L_{\alpha\underline{\gamma}}L_{\beta\underline{\delta}\zeta}$ $-2\times L_{\beta\underline{\gamma}}L_{\alpha\underline{\delta}\zeta}-4\times L_{\underline{\alpha}\zeta}L_{\underline{\beta\delta\gamma}}-L_{\alpha\beta\gamma\delta\zeta}$	$j\neq i, k\neq i$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(k)}u_\zeta^{(l)}}$	$+3\times\left(L_{\alpha\underline{\gamma\delta}}L_{\beta\underline{\zeta}}+L_{\beta\underline{\gamma\delta}}L_{\alpha\underline{\zeta}}-L_{\alpha\beta\underline{\gamma}}L_{\underline{\delta\zeta}}\right)$ $+2L_{\alpha\beta}L_{\gamma\delta\zeta}-L_{\alpha\beta\gamma\delta\zeta}$	i, j, k, l—different
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(j)}u_\gamma^{(k)}u_\delta^{(l)}u_\zeta^{(m)}}$	$-10\times L_{\underline{\alpha\beta\gamma}}L_{\underline{\delta\zeta}}-L_{\alpha\beta\gamma\delta\zeta}$	i, j, k, l, m—different
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}u_\zeta^{(i)}u_\mu^{(i)}}$	$8\left(15\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}L_{\underline{\zeta\mu}}\right)+4\left(15\times L_{\underline{\alpha\beta\gamma\delta}}L_{\underline{\zeta\mu}}\right)+2L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}u_\zeta^{(i)}u_\mu^{(j)}}$	$4\left(15\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\delta}}L_{\underline{\zeta}\mu}\right)+2\left(5\times L_{\underline{\alpha\beta\gamma\delta}}L_{\underline{\zeta}\mu}+10\times L_{\underline{\alpha\beta\gamma}\mu}L_{\underline{\delta\zeta}}\right)$ $+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}u_\zeta^{(j)}u_\mu^{(j)}}$	$8\left(3\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}L_{\zeta\mu}\right)+12\left(2\times L_{\alpha\beta}L_{\gamma\underline{\zeta}}L_{\delta\underline{\mu}}\right)+4\times L_{\underline{\alpha\beta\gamma}\zeta}L_{\underline{\delta}\mu}$ $+4\times L_{\underline{\alpha\beta\gamma}\mu}L_{\underline{\delta}\zeta}+4L_{\alpha\beta\gamma\delta}L_{\zeta\mu}+2\left(6\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\delta}\zeta\mu}\right)+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(j)}u_\zeta^{(j)}u_\mu^{(j)}}$	$4\left(9\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\underline{\delta}}}L_{\underline{\underline{\zeta\mu}}}\right)+6\times L_{\alpha\underline{\underline{\delta}}}L_{\underline{\beta\underline{\xi}}}L_{\underline{\gamma\underline{\mu}}}+9\times L_{\underline{\alpha\underline{\delta}}}L_{\underline{\beta\gamma\underline{\zeta\mu}}}$ $+2\left(3\times L_{\underline{\alpha\beta}}L_{\underline{\gamma}\delta\zeta\mu}+3\times L_{\alpha\beta\gamma\underline{\delta}}L_{\underline{\zeta\mu}}\right)+9\times L_{\underline{\alpha\beta\underline{\delta}}}L_{\underline{\gamma\underline{\zeta\mu}}}+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}u_\zeta^{(j)}u_\mu^{(k)}}$	$4\left(3\times L_{\alpha\underline{\beta}}L_{\underline{\gamma\delta}}L_{\zeta\mu}\right)+2\left(12\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\underline{\zeta}}}L_{\underline{\delta\underline{\mu}}}\right)+8\times L_{\underline{\alpha\underline{\zeta}}}L_{\underline{\beta\gamma\delta\underline{\mu}}}$ $+2\left(6\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\delta}\zeta\mu}\right)+2L_{\alpha\beta\gamma\delta}L_{\zeta\mu}+6\times L_{\underline{\alpha\beta}\zeta}L_{\underline{\gamma\delta}\mu}+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(j)}u_\zeta^{(j)}u_\mu^{(k)}}$	$6\times L_{\alpha\underline{\underline{\delta}}}L_{\underline{\beta\underline{\zeta}}}L_{\underline{\gamma}\mu}+4\left(3\times L_{\underline{\alpha\beta}}L_{\delta\zeta}L_{\underline{\gamma}\mu}\right)+2\left(6\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\underline{\delta}}}L_{\underline{\underline{\zeta}}\mu}\right)$ $+3\times L_{\underline{\alpha\beta}\delta\zeta}L_{\underline{\gamma}\mu}+6\times L_{\underline{\alpha\beta\underline{\delta}}\mu}L_{\underline{\gamma\underline{\zeta}}}+2L_{\gamma\delta}L_{\alpha\beta\zeta\mu}+2\times L_{\alpha\beta\gamma\underline{\delta}}L_{\underline{\zeta}\mu}$ $+2\left(3\times L_{\underline{\alpha\beta}}L_{\underline{\gamma}\delta\zeta\mu}\right)+6\times L_{\underline{\alpha\beta\underline{\delta}}}L_{\underline{\gamma\underline{\zeta}}\mu}+3\times L_{\underline{\alpha\beta}\mu}L_{\underline{\gamma}\delta\zeta}+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(j)}u_\zeta^{(k)}u_\mu^{(l)}}$	$6\times L_{\underline{\alpha\underline{\delta}}}L_{\underline{\beta\underline{\zeta}}}L_{\underline{\gamma\underline{\mu}}}+2\left(9\times L_{\underline{\alpha\beta}}L_{\underline{\gamma\underline{\delta}}}L_{\underline{\underline{\zeta\mu}}}\right)+3\times L_{\alpha\beta\gamma\underline{\delta}}L_{\underline{\zeta\mu}}$ $+9\times L_{\underline{\alpha\beta\underline{\delta\zeta}}}L_{\underline{\gamma\underline{\mu}}}+2\left(3\times L_{\underline{\alpha\beta}}L_{\underline{\gamma}\delta\zeta\mu}\right)+8\times L_{\underline{\alpha\beta\underline{\delta}}}L_{\underline{\gamma\underline{\zeta\mu}}}+L_{\alpha\beta\gamma\delta\zeta\mu}$	
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(j)}u_\zeta^{(k)}u_\mu^{(k)}}$	$8L_{\alpha\beta}L_{\gamma\delta}L_{\zeta\mu}+2\times\left(L_{\underline{\alpha}\gamma}L_{\underline{\beta}\delta}L_{\zeta\mu}+L_{\underline{\zeta}\alpha}L_{\underline{\mu}\beta}L_{\gamma\delta}+L_{\underline{\gamma}\zeta}L_{\underline{\delta}\mu}L_{\alpha\beta}\right)$ $8\times L_{\underline{\alpha\underline{\underline{\zeta}}}}L_{\underline{\beta\underline{\gamma}}}L_{\underline{\delta\underline{\underline{\mu}}}}+6\times L_{\underline{\alpha\beta\underline{\gamma}}}L_{\underline{\delta\underline{\underline{\zeta\mu}}}}+2\left(3\times L_{\underline{\alpha\beta\underline{\gamma\delta}}}L_{\underline{\underline{\underline{\zeta\mu}}}}\right)$ $+12\times L_{\underline{\alpha\beta\underline{\gamma\underline{\zeta}}}}L_{\underline{\delta\underline{\underline{\mu}}}}+4\times L_{\underline{\alpha\underline{\gamma}}\zeta}L_{\underline{\beta\underline{\delta}}\mu}+L_{\alpha\beta\gamma\delta\zeta\mu}$	

Table A1. (Continued.)

$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(j)}u_\zeta^{(k)}u_\mu^{(l)}}$	$4L_{\alpha\beta}L_{\gamma\delta}L_{\zeta\mu} + 2\left(4\times L_{\underline{\alpha\beta}}L_{\underline{\underline{\gamma}}\zeta}L_{\underline{\underline{\delta}}\mu}\right) + 10\times L_{\underline{\alpha}\underline{\underline{\underline{\zeta}}}}L_{\underline{\beta}\underline{\underline{\gamma}}}L_{\underline{\underline{\delta}}\underline{\underline{\underline{\mu}}}}$ $+2\left(2\times L_{\underline{\alpha\beta}}L_{\underline{\underline{\gamma\delta}}\zeta\mu}\right) + 13\times L_{\underline{\alpha}\underline{\underline{\beta\gamma}}\underline{\underline{\underline{\zeta}}}}L_{\underline{\underline{\delta}}\underline{\underline{\underline{\mu}}}} + 6\times L_{\underline{\alpha\beta}\underline{\underline{\gamma}}}L_{\underline{\delta}\underline{\underline{\underline{\zeta\mu}}}}$ $+4\times L_{\alpha\underline{\gamma}\underline{\underline{\zeta}}}L_{\beta\underline{\delta}\underline{\underline{\mu}}} + L_{\alpha\beta\gamma\delta\zeta\mu}$
$(W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(j)}u_\delta^{(k)}u_\zeta^{(l)}u_\mu^{(m)}}$	$2\left(3\times L_{\alpha\beta}L_{\gamma\underline{\delta}}L_{\underline{\zeta\mu}}\right) + 12\times L_{\underline{\alpha}\underline{\underline{\gamma}}}L_{\underline{\beta}\underline{\underline{\delta}}}L_{\underline{\underline{\zeta\mu}}} + 6\times L_{\alpha\beta\underline{\gamma}\underline{\delta}}L_{\underline{\zeta\mu}}$ $+2L_{\alpha\beta}L_{\gamma\delta\zeta\mu} + 8\times L_{\underline{\alpha}\underline{\underline{\gamma}}}L_{\underline{\beta}\underline{\underline{\delta}}\underline{\underline{\zeta\mu}}}$ $+4\times L_{\alpha\beta\underline{\gamma}}L_{\underline{\delta}\underline{\zeta\mu}} + 4\times L_{\alpha\underline{\gamma}\underline{\delta}}L_{\beta\underline{\zeta\mu}} + L_{\alpha\beta\gamma\ \delta\zeta\mu}$

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